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# Exact results on decoherence and entanglement in a system of N driven atoms and a dissipative cavity mode

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**Abstract.** We solve the dynamics of an open quantum system where N strongly driven two-level atoms are equally coupled on resonance to a dissipative cavity mode. Analytical results are derived on decoherence, entanglement, purity, atomic correlations and cavity field mean photon number. We predict decoherence-free subspaces for the whole system and the N-qubit subsystem, the monitoring of quantum coherence and purity decay by atomic populations measurements, the conditional generation of atomic multi-partite entangled states and of cavity cat-like states. We show that the dynamics of atoms prepared in states invariant under permutation of any two components remains restricted within the subspace spanned by the completely symmetric Dicke states. We discuss examples and applications in the cases $N = 3, 4$.

**PACS.** 42.50.Pq Cavity quantum electrodynamics; micromasers – 03.65.Yz Decoherence; open systems; quantum-statistical – 03.67.Bg Entanglement production and manipulation methods

## 1 Introduction

Cavity quantum electrodynamics (QED) concerns the interaction of atoms (or ions) with a quantized radiation field in a microwave or optical cavity [1,2,3]. The basic physical principles are quite well understood [4] and accurately tested [5,6,7]. Impressive advances in the experimental control on these systems allow to study fundamental issues in quantum physics such as entanglement and decoherence, related to the nonlocal correlations of composite systems at the microscopic level and to the boundary between the quantum and the classical descriptions. These issues have raised a huge interest due to the potentialities of the peculiar quantum behavior for applications in quantum information processing, communication, and computation [8] and the necessity to protect quantum coherence from noisy environments [9,10]. In this framework it is important to investigate nontrivial solvable models, representing somewhat idealized versions of systems implementable with the present cavity QED technology. In this paper we exactly solve the dynamics of an open multipartite system, where $N$ two-level atoms are equally coupled on resonance to a dissipative cavity mode and coherently driven by a strong external field. Cooled, trapped, deterministically loaded atoms [11,12,13] and trapped ion systems [14,15] in optical cavities, as well as Rydberg atoms crossing microwave cavities [1], appear as the most promising candidates for implementations.

We present a quite compact solution of the open system dynamics derived by phase-space techniques [16], used in previous works [17,18]. The N-atom subsystem can be described as a pseudo-spin system where the independent coupling of each atom to the cavity combines with the invariance of dynamics under permutation of any two atoms. A peculiarity of the system is that this description does not hold in the standard (energy or computational) basis, but in a rotated one. The permutational invariance reflects in the atomic coupling to the environment, leading to the existence of both global and atomic decoherence-free subspaces (DFS) [19,20]. In the latter case an initial $N$-qubit entanglement remains protected and available e.g. for quantum memories or quantum processors [21,22]. The structure of the general solution allows predicting a way to monitor the decay of quantum coherence and purity by measurements of atomic probabilities. In the limit of unitary dynamics these measurements can conditionally generate mesoscopic cat-like states of the cavity field. The preparation of atoms in states which are invariant under the exchange of any atom pair restricts the atomic dynamics to the subspace spanned by the completely symmetric Dicke states [23], some of which are genuinely multipartite entangled states [24]. This further simplifies the description of system dynamics in the important cases of atoms all prepared in a same state. Selected results for three and four atoms provide further insight on system and subsystems dynamics, including preservation and conditional generation of multipartite entanglement.

In Sect. 2 we introduce the model. The analytical solution of system dynamics is derived Sect. 3, where applications to $N = 3, 4$ are also reported. Results for transient and

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steady state regimes are discussed in Sect. 4, and multipartite entanglement protection is the object of Sect. 5. The main results are summarized in the Conclusions.

## 2 The physical model

We consider a set of $N$ two-level atoms interacting with a dissipative cavity field mode. The transition frequency $\omega_a$ between excited and ground states, $|e\rangle_l$ and $|g\rangle_l$ ($l = 1, ..., N$), is the same for all the $N$ atoms. A coherent external field of frequency $\omega_a$ simultaneously drives the atoms during the interaction with the cavity mode of frequency $\omega_f$ [25,26]. This kind of system is feasible in cavity QED experiments with two-level Rydberg atoms in a microwave cavity [1] or with three-level atoms effectively reduced to two levels interacting with an optical cavity [17], due to relevant advances recently achieved in cooling, trapping and deterministically loading atoms in optical cavities [11]. In both regimes atomic decays can be neglected, as we shall assume from now on. Similar dynamics could be also implemented by trapped ions interacting with a cavity mode [14].
The whole system Hamiltonian is

$$\hat{\mathcal{H}}(t) = \hbar\omega_f \hat{a}^\dagger\hat{a} + \hbar\sum_{l=1}^{N}\Big[\frac{\omega_a}{2}\hat{\sigma}_{z,l} + g(\hat{\sigma}_l^\dagger\hat{a} + \hat{\sigma}_l\hat{a}^\dagger) + \Omega(e^{-i\omega_a t}\hat{\sigma}_l^\dagger + e^{i\omega_a t}\hat{\sigma}_l)\Big], \tag{1}$$

where $\Omega$ is the Rabi frequency associated with the coherent driving field amplitude, $g$ the atom-cavity mode coupling constant taken equal for all atoms, $\hat{a}$ ($\hat{a}^\dagger$) the field annihilation (creation) operator, $\hat{\sigma}_l = |g\rangle_l\langle e|$ ($\hat{\sigma}_l^\dagger = |e\rangle_l\langle g|$) the atomic lowering (raising) operator, and $\hat{\sigma}_{z,l} = |e\rangle_l\langle e| - |g\rangle_l\langle g|$ the inversion operator.
In the perspective of experimental implementation of our scheme we add the effects of cavity mode dissipation, while we focus on resonance conditions ($\omega_a = \omega_f$) in order to derive an analytical solution of the system dynamics.
Therefore, we must solve the following master equation (ME) for the statistical density operator $\hat{\rho}_N'$ of the whole system

$$\dot{\hat{\rho}}_N' = -\frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{\rho}_N'] + \hat{\mathcal{L}}_f\hat{\rho}_N', \tag{2}$$

where

$$\hat{\mathcal{L}}_f\hat{\rho}_N' = \frac{k}{2}[2\hat{a}\hat{\rho}_N'\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}_N' - \hat{\rho}_N'\hat{a}^\dagger\hat{a}] \tag{3}$$

is the standard Liouville superoperator which describes the dissipative decay of the cavity field mode, with the rate $k$, due to the coupling to a thermal bath at zero temperature.
In the interaction picture the dissipative terms remain unchanged and the ME (2) can be rewritten as

$$\dot{\hat{\rho}}_N^I = -\frac{i}{\hbar}[\hat{\mathcal{H}}^I, \hat{\rho}_N^I] + \hat{\mathcal{L}}_f\hat{\rho}_N^I \tag{4}$$

where the Hamiltonian (1) has been replaced by the time-independent Hamiltonian $\hat{\mathcal{H}}^I = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_1$ with

$$\hat{\mathcal{H}}_0 = \hbar\Omega\sum_{l=1}^{N}\left(\hat{\sigma}_l^\dagger + \hat{\sigma}_l\right), \quad \hat{\mathcal{H}}_1 = \hbar g\sum_{l=1}^{N}\left(\hat{\sigma}_l^\dagger\hat{a} + \hat{\sigma}_l\hat{a}^\dagger\right). \tag{5}$$

Now we consider the unitary transformation $\hat{\mathcal{U}}(t) = e^{\frac{i}{\hbar}\hat{\mathcal{H}}_0 t}$ and we derive for the density operator $\hat{\rho}_N = \hat{\mathcal{U}}\hat{\rho}_N^I\hat{\mathcal{U}}^\dagger$ the following ME:

$$\dot{\hat{\rho}}_N = -\frac{i}{\hbar}[\hat{\mathcal{H}}_1', \hat{\rho}_N] + \hat{\mathcal{L}}_f\hat{\rho}_N \tag{6}$$

where the transformed Hamiltonian $\hat{\mathcal{H}}_1' = \hat{\mathcal{U}}\hat{\mathcal{H}}_1\hat{\mathcal{U}}^\dagger$ can be written as

$$\hat{\mathcal{H}}_1'(t) = \frac{\hbar g}{2}\hat{a}\sum_{l=1}^{N}\Big[(1 - e^{-2i\Omega t})\hat{\sigma}_l + (1 + e^{2i\Omega t})\hat{\sigma}_l^\dagger\Big] + \text{h.c.} \tag{7}$$

In the strong-driving regime for the interaction between the atoms and the external coherent field, $\Omega \gg g$, we can use the rotating-wave approximation obtaining the effective Hamiltonian [25,26]

$$\hat{\mathcal{H}}_{\text{eff}} = \frac{\hbar g}{2}(\hat{a} + \hat{a}^\dagger)\sum_{l=1}^{N}(\hat{\sigma}_l^\dagger + \hat{\sigma}_l) \tag{8}$$

We notice the presence of Jaynes-Cummings ($\hat{\sigma}_j^\dagger\hat{a} + \hat{\sigma}_j\hat{a}^\dagger$) as well as anti-Jaynes-Cummings ($\hat{\sigma}_j^\dagger\hat{a}^\dagger + \hat{\sigma}_j\hat{a}$) coupling terms of each coherently driven atom with the cavity field. Hereinafter we shall solve the master equation for the whole system density operator $\hat{\rho}_N(t)$

$$\dot{\hat{\rho}}_N = -\frac{i}{\hbar}[\hat{\mathcal{H}}_{\text{eff}}, \hat{\rho}_N] + \hat{\mathcal{L}}_f\hat{\rho}_N. \tag{9}$$

## 3 Analytical solution and system dynamics

In order to solve the general $N$-atom problem we introduce the collective atomic operator $\hat{S}_x = \frac{1}{2}\sum_{l=1}^{N}\hat{\sigma}_{x,l} = \frac{1}{2}\sum_{l=1}^{N}(\hat{\sigma}_l^\dagger + \hat{\sigma}_l)$, so that the effective Hamiltonan assumes the simple form

$$\hat{\mathcal{H}}_{\text{eff}} = \hbar g(\hat{a} + \hat{a}^\dagger)\hat{S}_x. \tag{10}$$

We recall that the eigenstates of the spin operator $\hat{\sigma}_{x,l}$ are the rotated states $|\pm\rangle_l = \frac{|g\rangle_l \pm |e\rangle_l}{\sqrt{2}}$ where $\hat{\sigma}_{x,l}|\pm\rangle_l = \lambda_l^\pm|\pm\rangle_l$ with $\lambda_l^\pm = \pm1$. For the whole atomic subspace we consider the basis of $2^N$ states $\{|i\rangle_N\}$ where any $|i\rangle_N$ is an eigenstate of the collective spin operator $\hat{S}_x$. The corresponding eigenvalue $s_i = (1/2)\sum_{l=1}^{N}\lambda_l^\pm$ is half the difference between the number of $|+\rangle$ and $|-\rangle$ components of state $|i\rangle_N$, regardless of the exchange of any qubit pair, and it

can assume $N+1$ values from $-N/2$ to $N/2$ with steps $|\Delta s_i| = 1$. We notice that the eigenvalues $s_i$ have a degeneracy order given by $n(s_i) = \frac{N!}{(N/2+s_i)!(N/2-s_i)!}$ that is greater than one if $-N/2 < s_i < N/2$.

The general solution of the ME (9) can be derived by introducing the decomposition of the density operator $\hat{\rho}_N(t) = \sum_{i,j=1}^{2^N} {}_N\langle i|\hat{\rho}_N(t)|j\rangle_N |i\rangle_N\langle j|$ on the N-atom rotated basis $|i\rangle_N$, so that it is equivalent to the following set of $2^{2N}$ uncoupled evolution equations for the field operators $\hat{\rho}_{N,ij} = {}_N\langle i|\hat{\rho}_N(t)|j\rangle_N$

$$\begin{aligned}\dot{\hat{\rho}}_{N,ij} = &-ig\big[s_i(\hat{a}+\hat{a}^\dagger)\hat{\rho}_{N,ij} - s_j\hat{\rho}_{N,ij}(\hat{a}+\hat{a}^\dagger)\big] \\ &+ \hat{\mathcal{L}}_f\hat{\rho}_{N,ij}.\end{aligned} \tag{11}$$

Equation (11) can be solved by a combination of phase space techniques [16] with the method of characteristics [27]. Starting from the cavity in the vacuum state $|0\rangle$ and the atoms in any pure state

$$|\Psi(0)\rangle_N = |0\rangle \otimes \sum_{i=1}^{2^N} c_{N,i}|i\rangle_N \tag{12}$$

with the normalization condition $\sum_{i=1}^{2^N}|c_{N,i}|^2 = 1$, we obtain the compact solution for the whole system density operator

$$\begin{aligned}\hat{\rho}_N(t) = &\sum_{i,j=1}^{2^N} c_{N,i}c^*_{N,j}[f(t)]^{(s_i-s_j)^2} \times \\ &\times |-2s_i\alpha(t)\rangle\langle -2s_j\alpha(t)| \otimes |i\rangle_N\langle j|.\end{aligned} \tag{13}$$

We see that the dynamics correlates the eigenstates of $\hat{S}_x$ with cavity field coherent states of amplitude proportional to

$$\alpha(t) = i\frac{g}{k}\left(1 - e^{-\frac{k}{2}t}\right). \tag{14}$$

The one-atom decoherence function

$$f(t) = f_1(t)e^{2|\alpha(t)|^2} = e^{-\frac{2g^2}{k}t + \frac{4g^2}{k^2}\left(1-e^{-\frac{k}{2}t}\right)} e^{2|\alpha(t)|^2} \tag{15}$$

naturally splits into two parts: $f_1(t)$, which will appear in the atomic subsystem dynamics, and $e^{2|\alpha(t)|^2}$, that is the field states normalization. It is responsible for the decay of coherences and depends on the dimensionless parameters $(g/k)^2$ and $kt$.

In order to evaluate the degree of mixedness of the state $\hat{\rho}_N(t)$ we derive from eq. (13) the purity

$$\begin{aligned}\mu_N(t) &= Tr[\hat{\rho}^2_N(t)] \\ &= \sum_{i,j=1}^{2^N} |c_{N,i}|^2|c_{N,j}|^2[f(t)]^{2(s_i-s_j)^2}.\end{aligned} \tag{16}$$

A remarkable consequence of the general solution of eq. (13) is the existence of a global DFS for any even value of the number N of atoms, when the eigenvalue $s_i$ can assume the value zero. In this case there is no time evolution for the initial states of eq. (12) containing only the corresponding $n(0) = N!/[(N/2)!]^2$ atomic eigenstates $|i\rangle_N$. Let us consider for example the case of $N = 4$ atoms. The DFS is spanned by the tensor product of the cavity vacuum state and $n(0) = 6$ states $|i\rangle_4$ with the same number of $|+\rangle$ and $|-\rangle$ components (see Table 1). It preserves any initial global state within this subspace, protecting any entangled atomic preparation for quantum information purposes.

Another interesting feature of the case with even N follows from the presence in eq. (13) of terms with the cavity field in the vacuum state. Namely, if the optical cavity field is accessible to measurements, the absence of a response by an on/off detector generates a pure N-qubit state, that can be a multipartite entangled state. We consider again the $N = 4$ atoms case. Starting e.g. from the four atoms prepared in the ground state, the density operator of eq. (13) contains a time-independent part $(\sqrt{6}/4)(|0\rangle \otimes |\Psi\rangle_a)$, where $|\Psi\rangle_a = (1/\sqrt{6})(|++--\rangle + |+-+-\rangle + |+--+\rangle + |-++-\rangle + |-+-+\rangle + |--++\rangle)$. Hence a null measurement of the optical cavity field generates the pure 4-qubit state $|\Psi\rangle_a$ whose entanglement properties will be discussed later.

### 3.1 Completely symmetric Dicke states

In order to exploit the system dynamical invariance under exchange of any atom pair, we consider initial states (eq. (12)) having in the atomic part only symmetric states or symmetrized combinations of states $|i\rangle_N$. In this case the atomic part of $\hat{\rho}_N(t)$ remains confined in the subspace spanned by only $N+1$ (instead of $2^N$) states that we denote as $|\frac{N}{2}, s\rangle$, where $-\frac{N}{2} \le s \le \frac{N}{2}$ with steps $|\Delta s| = 1$. We notice that the above states are analogous to the so-called completely symmetric Dicke states (CSD) in [23], written in the energy basis instead of the rotated one. For instance for $N = 4$ the states $|2, s\rangle$ with $-2 \le s \le 2$ are the symmetrized combinations of the states listed in Table 1. All the previous treatment can be adapted correspondingly. In particular, starting from any superposition of CSD states

$$|\Psi(0)\rangle_N = |0\rangle \otimes \sum_{s=-N/2}^{N/2} b_{N,s}|\frac{N}{2}, s\rangle \tag{17}$$

with the normalization condition $\sum_{s=-N/2}^{N/2}|b_{N,s}|^2 = 1$, the general solution (13) can be rewritten as

$$\begin{aligned}\hat{\rho}_N(t) = &\sum_{s,s'=-N/2}^{N/2} b_{N,s}b^*_{N,s'}[f(t)]^{(s-s')^2} \times \\ &\times |-2s\alpha(t)\rangle\langle -2s'\alpha(t)| \otimes |\frac{N}{2}, s\rangle\langle\frac{N}{2}, s'|.\end{aligned} \tag{18}$$

In this case the interaction correlates cavity field coherent states with atomic CSD states. These results include the important case of all atoms prepared in the ground state,

**Table 1.** Atomic DFSs for $N = 4$ qubits.

| $\mathbf{s_i}$ | $\mathbf{n(s_i)}$ | $\lvert\mathbf{i}\rangle_4$ |
|---|---|---|
| 2 | 1 | $\{\lvert ++++\rangle\}$ |
| 1 | 4 | $\{\lvert +++-\rangle, \lvert ++-+\rangle, \lvert +-++\rangle, \lvert -+++\rangle\}$ |
| 0 | 6 | $\{\lvert ++--\rangle, \lvert +-+-\rangle, \lvert +--+\rangle, \lvert -+-+\rangle, \lvert -++-\rangle, \lvert --++\rangle\}$ |
| -1 | 4 | $\{\lvert ---+\rangle, \lvert --+-\rangle, \lvert -+--\rangle, \lvert +---\rangle\}$ |
| -2 | 1 | $\{\lvert ----\rangle\}$ |

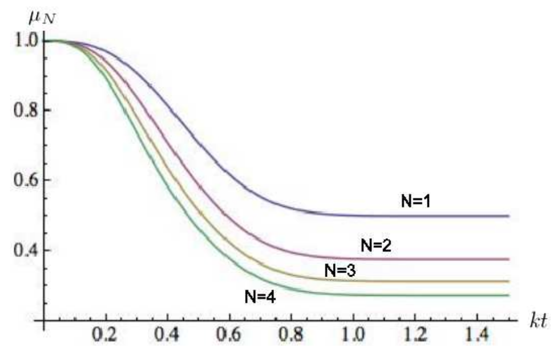


**Fig. 1.** Time evolution of the purity of the whole system for different values of $N$ and for the fixed dimensionless parameter $g/k = 5$.

where $b_{N,s} = (1/\sqrt{2^N})n(s)$. In another relevant case, with all atoms prepared in the excited state, the only change in (18) is the replacement $f(t) \to -f(t)$. In such cases the purity (16) reduces to

$$\mu_N(t) = \frac{1}{2^{2N}} \sum_{s,s'=-N/2}^{N/2} f(t)^{2(s-s')^2} \tag{19}$$

whose asymptotic value can be written in a closed form in terms of the gamma function $\Gamma$

$$\mu_N^{SS} = \frac{1}{2^{2N}} \sum_{s=-N/2}^{N/2} n^2(s) = \frac{1}{2^{2N}} \sum_{l=0}^{N} \binom{N}{l}^2 = \frac{\Gamma(N+\frac{1}{2})}{\sqrt{\pi}\Gamma(N+1)} \tag{20}$$

where $l \equiv s + N/2$ and $\binom{N}{l}$ is the binomial coefficient. In Fig. 1 we show the time evolution of the system purity (19) for a fixed value of $g/k$ and different qubit numbers $N = 1, ..., 4$, where the steady state values are $\mu_1^{SS} = 1/2$, $\mu_2^{SS} = 3/8$, $\mu_3^{SS} = 5/16$, $\mu_4^{SS} = 35/128$. The greater the value of $N$, the faster the decay of the global coherences. Varying the ratio $g/k$ instead of $N$, the asymptotic behavior does not change whereas the decay is faster (slower) for increasing (decreasing) values of $g/k$.
We remark that the atomic preparation in one of the CSD states is equivalent to the qubit encoding in the corresponding DFS. In particular, for even values of $N$, the CSD atomic state with $s = 0$ and the cavity in the vacuum state belong to a global DFS.

## 3.2 Subsystem dynamics

We derive some general results on the cavity mode and the atomic subsystems, providing the time-dependent expressions of the corresponding reduced density operators and purities, as well as of the mean number of photons in the cavity.
If we trace the whole system density operator of eq. (13) over the atomic variables, we find the expression for the reduced density operator of the cavity field

$$\hat{\rho}_{N,f}(t) = \sum_{i=1}^{2^N} |c_{N,i}|^2 |-2s_i\alpha(t)\rangle\langle -2s_i\alpha(t)| \tag{21}$$

that is a statistical mixture of coherent states, and whose purity is

$$\mu_{N,f}(t) = Tr[\hat{\rho}_{N,f}^2(t)] = \\ = \sum_{i,j=1}^{2^N} |c_{N,i}|^2 |c_{N,j}|^2 [e^{2|\alpha(t)|^2}]^{2(s_i-s_j)^2}. \tag{22}$$

From the density operator $\hat{\rho}_{N,f}(t)$ we can derive the expression of the mean photon number

$$\langle \hat{a}^\dagger \hat{a}\rangle(t) = Tr_f[\hat{\rho}_{N,f}(t)\hat{a}^\dagger\hat{a}] = 4|\alpha(t)|^2 \sum_{i=1}^{2^N} s_i^2 |c_{N,i}|^2. \tag{23}$$

In the case of all atoms in the ground state, $c_{N,i} = 1/\sqrt{2^N}$, one obtains

$$\langle \hat{a}^\dagger \hat{a}\rangle(t) = \frac{|\alpha(t)|^2}{2^{N-2}} \sum_{l=0}^{N} \left(l - \frac{N}{2}\right)^2 \binom{N}{l} = N|\alpha(t)|^2 \tag{24}$$

showing that each atom gives the same average contribution to the cavity field.
By tracing the whole system density operator over the field variables, we obtain the reduced atomic density operator

$$\hat{\rho}_{N,a}(t) = \sum_{i,j=1}^{2^N} c_{N,i} c_{N,j}^* [f_1(t)]^{(s_i-s_j)^2} |i\rangle_N \langle j|. \tag{25}$$

We notice that if the atoms are prepared in any superposition of eigenstates $|i\rangle_N$ corresponding to a degenerate eigenvalue $-N/2 < s_i < N/2$, the state does not evolve. Therefore we identify $N-1$ atomic DFSs with dimension

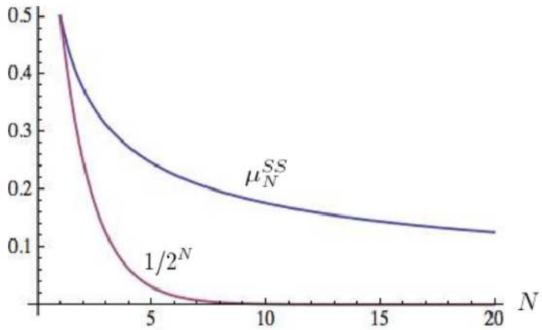


**Fig. 2.** Comparison between the $N$-qubit purity $\mu_{N,a}(t)$ in (26) evaluated at steady state and $1/2^N$ that is the purity of a maximally mixed state.

$n(s_i)$ greater than one, where an initial entanglement can be protected. The purity of state (25) is

$$\mu_{N,a}(t) = Tr[\hat{\rho}^2_{N,a}(t)] = \\ = \sum_{i,j=1}^{2^N} |c_{N,i}|^2 |c_{N,j}|^2 [f_1(t)]^{2(s_i - s_j)^2}. \quad (26)$$

The decay of atomic purity ruled by $f_1(t)$ is faster than the global purity decay (16), ruled by $f(t)$. However the asymptotic behavior is the same and we can use the result (20) in order to make a comparison with the case of maximally mixed states, whose purity is equal to $1/2^N$. In Fig. 2 we see that the state is maximally mixed only for $N = 1$, where actually the atom becomes maximally entangled with cavity field [17]. For any $N > 1$ the state is never maximally mixed due to the survival of coherences in the DFSs. Also we notice that the field purity (22) remains slightly larger than the atomic one because the decoherence function $f_1(t)$ is replaced by a non-vanishing exponential function.

As an application of the atomic subsystem dynamics we rewrite the atomic density matrix eq. (25) in the standard basis for the case $N = 3$. Starting, for instance, from the three atoms in the ground state the diagonal matrix elements provide the following joint probabilities for the atomic level populations

$$P_{eee}(t) = \frac{1}{32}\Big[10 - 15 f_1(t) + 6 f_1^4(t) - f_1^9(t)\Big] \quad (27a)$$

$$P_{eeg}(t) = \frac{1}{32}\Big[2 - f_1(t) - 2 f_1^4(t) + f_1^9(t)\Big] \quad (27b)$$

$$P_{egg}(t) = \frac{1}{32}\Big[2 + f_1(t) - 2 f_1^4(t) - f_1^9(t)\Big] \quad (27c)$$

$$P_{ggg}(t) = \frac{1}{32}\Big[10 + 15 f_1(t) + 6 f_1^4(t) + f_1^9(t)\Big] \quad (27d)$$

where eqs. (27b) and (27c) represent one third of the probability to detect, respectively, two atoms in the excited state or in the ground state, independently from the atomic ordering. The three-atom probabilities (27) are shown in Fig. 3.

We can see that at steady state the joint probability that three atoms are in the same state is equal to $5/16$, that is an atomic correlation (bunching) effect, whereas the joint probability of the other two outcomes is $3/16$, showing an antibunching effect. By exploiting the above expressions (27) it is possible to monitor the two-atom decoherence function $f_1^4(t)$ measuring the sums $P_{eee}(t) + P_{ggg}(t)$ or $P_{eeg}(t) + P_{egg}(t)$. Remarkably the N-qubit decoherence originates from the one-atom decoherence function [17] which can be monitored by atomic population measurements via the relation $f_1(t) = P_g(t) - P_e(t)$, as well as the N-qubit purity according to (26). In that case the atom and the field can approach maximally entangled states in the limits $kt \ll 1$ and $(g/k)^2 \gg 1$, and the entanglement (measured by the Von Neumann subsystem entropy) is also described by $f_1(t)$.

## 4 Transient and steady state results

In the Hamiltonian limit, $kt \ll 1$, of small cavity decay rate and/or short interaction times, $f(t) \to 1$, $\alpha(t) \to \tilde{\alpha}(t) \equiv i\frac{gt}{2}$ and $\rho_N(t) \to |\tilde{\Psi}(t)\rangle_N \langle \tilde{\Psi}(t)|$ where the global cat-like state

$$|\tilde{\Psi}(t)\rangle_N = \sum_{i=1}^{2^N} c_{N,i} |-2 s_i \tilde{\alpha}(t)\rangle \otimes |i\rangle_N. \quad (28)$$

As an example, for $N = 3$ we consider the generation of the pure state $|\tilde{\Psi}(t)\rangle_3$. Starting from three atoms in the ground state, so that $c_{3,i} = 1/\sqrt{8}$, we obtain for $kt \ll 1$ an evolved state that we rewrite in the standard atomic basis

$$\begin{aligned} |\tilde{\Psi}\rangle_3 = \frac{1}{\sqrt{8}}\Big[&(|-3\tilde{\alpha}\rangle - 3|-\tilde{\alpha}\rangle + 3|\tilde{\alpha}\rangle - |3\tilde{\alpha}\rangle) \otimes |eee\rangle \\ &+ (|-3\tilde{\alpha}\rangle + 3|-\tilde{\alpha}\rangle + 3|\tilde{\alpha}\rangle + |3\tilde{\alpha}\rangle) \otimes |ggg\rangle \\ &+ (|-3\tilde{\alpha}\rangle - |-\tilde{\alpha}\rangle - |\tilde{\alpha}\rangle + |3\tilde{\alpha}\rangle) \otimes (|eeg\rangle + |ege\rangle + |gee\rangle) \\ &+ (|-3\tilde{\alpha}\rangle + |-\tilde{\alpha}\rangle - |\tilde{\alpha}\rangle - |3\tilde{\alpha}\rangle) \otimes (|egg\rangle + |geg\rangle + |gge\rangle)\Big] \end{aligned} \quad (29)$$

where for brevity we have defined $\tilde{\alpha} \equiv \tilde{\alpha}(t)$. We notice a superposition of mesoscopic cat-like states of the cavity field correlated with atomic states with the same number of ground (or excited) atoms, which are two fully separable and two entangled 3-qubit states (a W and an inverted-W state) [28]. An interesting consequence of eq. (29) is that a simultaneous detection of the three atoms in any state prepares the cavity field in the corresponding cat-like state. In Fig. 4 we show the Wigner function that describes in phase space the cat-like state generated for atomic detections in the ground state.

After the transient the coupling of the field to the environment introduces in the solution (13) the field-atoms coherences $f(t), f^4(t), ..., f^{N^2}(t)$. Note that these powers of the decoherence function can be obtained by the substitution $g \to Ng$, which exactly reflects the independent interaction of each atom with the cavity field.

In the steady state limit $kt \gg 1$ the density operator $\hat{\rho}_N(t)$

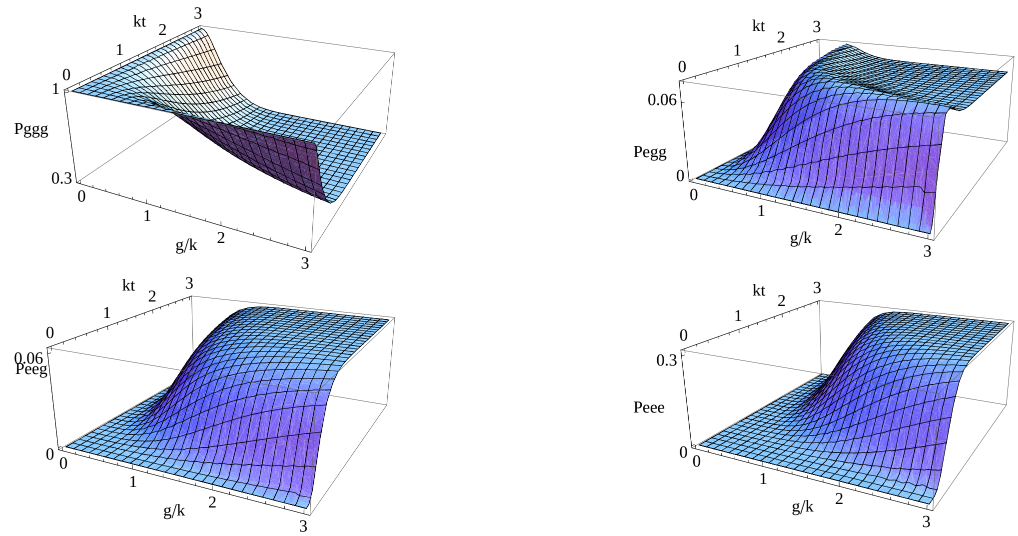


**Fig. 3.** Three-atom joint probabilities vs dimensionless coupling constant $g/k$ and time $kt$ from eqs. (27).

becomes a statistical mixture of the pure states superimposed in the global cat-like state (28) generated in the transient

$$\hat{\rho}_N^{SS} = \sum_{i=1}^{2^N} |c_{N,i}|^2 |-2s_i\alpha^{SS}\rangle\langle -2s_i\alpha^{SS}| \otimes |i\rangle_N\langle i|, \quad (30)$$

where $\alpha^{SS} = i\frac{g}{k}$. The system (subsystem) purity at steady state was discussed in the previous section.

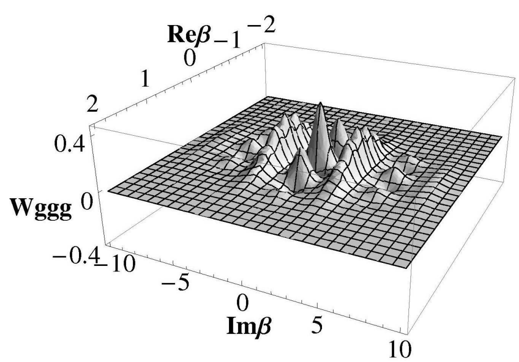


**Fig. 4.** Wigner function $W_{ggg}$ of the cavity field state conditioned to the detection of the three atoms in the ground state, for parameters values $kt = 0.05$ and $g/k = 110$.

## 5 Protection of multipartite entanglement

Now we recall some concepts and tools in order to analyze the multipartite entanglement properties of some atomic states encoded in DFSs. The 3-tangle measure introduced in [29] evaluates the amount of entanglement shared by all the three qubits through the quantity $\tau_{123} = C_{12}^2 + C_{13}^2 - C_{1(23)}^2$, where $C_{ij}$ is the concurrence of the qubit pair $(i,j)$. A generalization to the case of N qubits (with N even) was given in [30] by the $N$-tangle measure defined as $\tau_N = |\langle\psi|\tilde{\psi}\rangle|^2$ with $|\tilde{\psi}\rangle = \sigma_y^{\otimes N}|\psi^*\rangle$, where $|\psi\rangle$ is the generic $N$-qubit state, $|\psi^*\rangle$ its complex conjugate and $\sigma_y$ one of the Pauli matrices. Another useful tool is the residual bipartite entanglement measure (see [28]) which evaluates the robustness of entanglement against the loss of information; this measure is provided, for instance, by the average squared concurrence $\overline{C^2}$ calculated for any two residual qubits when the other $N-2$ are traced out.
Following the previous analysis concerning the DFSs we first consider the atomic subsystem for $N = 3$. In this case we find that the three qubits do not evolve in time if they are prepared in any of the four decoherence-free CSD states

$$|3/2, 3/2\rangle = |+++\rangle \quad (31a)$$

$$|3/2, 1/2\rangle = \frac{|++-\rangle + |+-+\rangle + |-++\rangle}{\sqrt{3}} \quad (31b)$$

$$|3/2, -1/2\rangle = \frac{|+--\rangle + |-+-\rangle + |--+\rangle}{\sqrt{3}} \quad (31c)$$

$$|3/2, -3/2\rangle = |---\rangle. \quad (31d)$$

Two of them, (31a) and (31d), are manifestly separable. The other two states, (31b) and (31c), show interesting entanglement properties. They have no full tripartite entanglement ($\tau_{123} = 0$) according to the 3-tangle measure. However each qubit pair retains the maximal residual bipartite entanglement $\overline{C^2} = 4/9$. These kind of states show a multipartite entanglement characteristic of W-like states.
Let us now investigate the dynamics of four qubits which presents five decoherence-free CSD states, including two separable states $|2, \pm 2\rangle$, and the multipartite entangled states $|2, \pm 1\rangle$ and $|2, 0\rangle$. The relevance for applications in

quantum information processing is that the state $|2,0\rangle$ turns out to be maximally entangled according to the 4-tangle measure ($\tau_4 = 1$), whereas the states $|2,\pm1\rangle$ have no four-partite entanglement ($\tau_4 = 0$), but each of them exhibits an equal maximal reduced bipartite entanglement, $\overline{C^2} = 1/4$ (W-like states), by tracing over any qubit pair. We remark that all CSD states of the type $|N/2,\tilde{s}\rangle$ with $\tilde{s} = \pm(N-2)/2$ have entanglement properties similar to that of states $|W_N\rangle$ introduced in [28]. By tracing the atomic density operators $|N/2,\tilde{s}\rangle\langle N/2,\tilde{s}|$ over any $N-2$ parties we always obtain the reduced density operators for the bipartite system

$$\rho_{\pm} = \frac{1}{N}\left(2|\Phi^-\rangle\langle\Phi^-| + (N-2)|\pm\pm\rangle\langle\pm\pm|\right). \tag{32}$$

Hence for the average squared concurrence we simply obtain the value $\overline{C^2} = (2/N)^2$.
We further notice that we can rewrite the states $|2,\pm1\rangle$ and $|2,\pm0\rangle$ as

$$\begin{aligned}
|2,\pm1\rangle &= \frac{1}{\sqrt{2}}(|\pm\pm\rangle_{12}|\Phi^-\rangle_{34} + |\Phi^-\rangle_{12}|\pm\pm\rangle_{34}) \\
|2,0\rangle &= \frac{1}{\sqrt{6}}\Big(|\Phi^-\rangle_{12}|\Phi^-\rangle_{34} + |\Phi^-\rangle_{13}|\Phi^-\rangle_{24} + \\
&\quad + |\Phi^-\rangle_{14}|\Phi^-\rangle_{23}\Big)
\end{aligned} \tag{33}$$

thus generalizing the results derived in [18] where we showed that the two-atom maximally entangled Bell state $|\Phi^-\rangle_{ij} = \frac{1}{\sqrt{2}}(|+-\rangle_{ij} + |-+\rangle_{ij})$ and the two separable states $|\pm\pm\rangle$ do not evolve in time.
Another interesting application is to encode the four qubits in some states of a special basis called Bell gem [31], which is a generalization of the well known Bell basis $|\Phi^{\pm}\rangle = (1/\sqrt{2})(|gg\rangle \pm |ee\rangle)$ and $|\Psi^{\pm}\rangle = (1/\sqrt{2})(|ge\rangle \pm |eg\rangle)$. It is composed by maximally entangled states, according to the 4-tangle measure ($\tau_4 = 1$), which can be obtained by simple quantum logic circuits starting from four unentangled qubits in the computational basis. Let us consider the cavity field prepared in the vacuum state $|0\rangle$ and the four qubits in one of the last three elements of the Bell gem $(1/\sqrt{2})(|\Phi^+\Psi^+\rangle - |\Psi^+\Phi^+\rangle)$, $(1/\sqrt{2})(|\Psi^-\Phi^-\rangle \pm |\Phi^-\Psi^-\rangle)$. The whole system does not evolve in time because these three initial atomic states belong to the DFS corresponding to the eigenvalue $s_i = 0$, thus maintaining the maximum multipartite entanglement in the atomic subsystem.

## 6 Conclusions

We have solved the dynamics of a feasible cavity QED system where $N$ strongly driven two-level atoms are equally and resonantly coupled to an optical field mode in contact with an environment and initially in the vacuum state. For negligible atomic decay we have derived a compact solution of the open system master equation in terms of coherent field states, atomic pseudo-spin states, and suitable decoherence functions. We have derived and discussed a number of exact results on system and subsystems dynamics which are also of interest for quantum information applications, including decoherence-free subspaces and multipartite ($N$-qubit) entanglement protection. In addition we have suggested a way to monitor decoherence by atomic population measurements. For atoms prepared in symmetric states with respect to the exchange of any atom pair, including the physically important preparation in the same (ground or excited) state, the dynamics is entirely expressed in terms of symmetric Dicke states. Applications in the cases with $N = 3, 4$ have been discussed.

## References

1. S. Haroche, J. Raimond, *Exploring the Quantum* (Oxford University Press, 2006)
2. H. Mabuchi, A.C. Doherty, Science **298**, 1372 (2002)
3. C. Monroe, Nature **416**, 238 (2002)
4. E. Jaynes, F. Cummings, Proc. IEEE **51**, 89 (1963)
5. D. Meschede, H. Walther, G. Müller, Phys. Rev. Lett. **54**, 551 (1985)
6. M. Brune, F. Schmidt-Kaler, A. Maali, J. Dreyer, E. Hagley, J.M. Raymond, S. Haroche, Phys. Rev. Lett. **76**, 1800 (1996)
7. J. McKeever, A. Boca, A.D. Boozer, J.R. Buck, H.J. Kimble, Nature **425**, 268 (2003)
8. M.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000)
9. E. Joos, H.D. Zeh, C. Kiefer, D. Giulini, J. Kupsch, I.O. Stamatescu, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer, Berlin, 1996)
10. W.H. Zurek, Rev. Mod. Phys. **75**, 715 (2003)
11. S. Nussman, G.S. Agarwal, K. Murr, M. Hijlkema, B. Weber, A. Kuhn, G. Rempe, Nature Phys. **1**, 122 (2005)
12. A.D. Boozer, A. Boca, R. Miller, T.E. Northup, H.J. Kimble, Phys. Rev. Lett. **97**, 083602 (2006)
13. K.M. Fortier, S.Y. Kim, M.J. Gibbons, P. Ahmadi, M.S. Chapman, Phys. Rev. Lett. **98**, 233601 (2007)
14. A.B. Mundt, A. Kreuter, C. Becher, D. Leibfried, J. Eschner, F. Schmidt-Kaler, R. Blatt, Phys. Rev. Lett. **89**, 103001 (2002)
15. M. Keller, B. Lange, K. Hayasaka, W. Lange, H. Walther, Nature **431**, 1075 (2004)
16. K.E. Cahill, R.J. Glauber, Physical Review **177**, 1882 (1969)
17. P. Lougovski, F. Casagrande, A. Lulli, E. Solano, Phys. Rev. A **76**, 033802 (2007)
18. M. Bina, F. Casagrande, A. Lulli, E. Solano, Phys. Rev. A **77**, 033839 (2008)
19. L.M. Duan, G.C. Guo, Phys. Rev. Lett. **79**, 1953 (1997)
20. P. Zanardi, M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997)
21. D.L. Moehring, M.J. Madsen, R.N. Kohn, L.M. Duan, C. Monroe, J. Opt. Soc. Am. B **14**, 300 (2007)
22. P. Zoller, *et al.*, Eur. Phys. J. D **36**, 203 (2005)
23. L. Mandel, E. Wolf, *Optical coherence and quantum optics* (Cambridge University Press, Cambridge, 1995)
24. C. Thiel, J. von Zanthier, T. Bastin, E. Solano, G.S. Agarwal, Phys. Rev. Lett. **99**, 193602 (2007)
25. E. Solano, G. Agarwal, H. Walther, Phys. Rev. Lett. **90**, 027903 (2003)
26. P. Lougovski, E. Solano, H. Walter, Phys. Rev. A **71**, 013811 (2005)

27. S.M. Barnett, P.M. Radmore, *Methods in Theoretical Quantum Optics* (Clarendon Press, Oxford, 1997)
28. W. Dür, G. Vidal, J.I. Cirac, Phys. Rev. A **62**, 062314 (2000)
29. V. Coffman, J. Kundu, W.K. Wootters, Phys. Rev. A **61**, 052306 (2000)
30. A. Wong, N. Christensen, Phys. Rev. A **63**, 044301 (2001)
31. G. Jaeger, *Quantum Information* (Springer, 2007)